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On the geometry and homology of certain simple stratified varieties

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Abstract

We study certain mild degenerations of algebraic varieties which appear in the analysis of a large class of supersymmetric theories, including superstring theory. We analyze Witten's σ-model [Nucl. Phys. B 403 (1993) 159] and find that the non-transversality of the superpotential induces additional singularities and a stratification of the ground state variety. This stratified variety admits certain homology groups such that $\oplus_q H^{2q}$ satisfies the "Kähler package" of requirements [Ann. Math. Studies 102 (1982) 303]. Also, this $\oplus_q H^{2q}$ extends the "flopped" pair of small resolutions [Nucl. Phys. B 416 (1994) 414; Nucl. Phys. B 330 (1990) 49; Commun. Math. Phys. 119 (1988) 431] to an "(exo)flopped" triple, and is compatible with both mirror symmetry [S.-T. Yau (Ed.), Mirror Manifolds, International Press, Hong Kong, 1990; B. Greene, S.-T. Yau (Eds.), Mirror Manifolds II, International Press, Hong Kong, 1996] and string theory [Mod. Phys. Lett. A 12 (1997) 521; Nucl. Phys. B 451 (1995) 96] results. Finally, we revisit the conifold transition [Nucl. Phys. B 330 (1990) 49] as it applies in our formalism.

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1. Introduction, results and summary

In string theory, rather than being an assumed arena, the spacetime is identified with the dynamically determined 'ground state variety' of a (supersymmetric) σ-model [11,20,23].

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In the simplest physically interesting and non-trivial case [3,23], the spacetime is of the form $M^{3,1} \times K$, where K is a compact Calabi-Yau three-fold modeled from the (bosonic subset of the) 'field space' of the σ-model[1], $\mathcal{F} = \{p, s_0, \ldots, s_4\} \simeq \mathbb{C}^6$, which admits a $\mathbb{C}^*$ action:

$$\hat{\lambda} : \{p, s_0, \ldots, s_4\} \mapsto \{\lambda^{-5} p, \lambda s_0, \ldots, \lambda s_5\}, \quad \lambda \in \mathbb{C}^*. \tag{1}$$

The 'ground state variety' is defined to be [2,23]

$$\mathcal{V} \stackrel{\text{def}}{=} [(\partial W)^{-1}(0) - 0]/\hat{\lambda}, \tag{2}$$

with the $\hat{\lambda}$-invariant holomorphic 'superpotential'

$$W \stackrel{\text{def}}{=} p \cdot G(s). \tag{3}$$

Alternatively, we denote by $\hat{\hat{\lambda}}$ the $|\lambda| = 1$ restriction of the map (1), and define the 'potential'

$$U_r \stackrel{\text{def}}{=} \|\partial W\|^2 + D_r^2, \tag{4}$$

where

$$D_r \stackrel{\text{def}}{=} \|s\|^2 - 5|p|^2 - r, \quad r \in \mathbb{R}. \tag{5}$$

Then

$$\mathcal{V} \simeq [U_r^{-1}(0) - 0]/\hat{\hat{\lambda}}. \tag{6}$$

Due to the positive definiteness of U_r,

$$U_r^{-1}(0) = (\partial W)^{-1}(0) \cap D_r^{-1}(0). \tag{7}$$

Furthermore, the $\hat{\lambda}$-invariance of $W = pG$ implies that $G(s)$ is a degree-5 homogeneous complex polynomial

$$G(\lambda s_0, \ldots, \lambda s_4) = \lambda^5 G(s_0, \ldots, s_4), \tag{8}$$

whereupon the zero locus of ∂W is the intersection of the cones

$$(\partial W)^{-1}(0) = G^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0). \tag{9}$$

The above definition may then be rephrased as follows.

Definition 1. Given the polynomials $G(s)$ and D_r as defined in Eqs. (8) and (5), respectively, the 'ground state variety' is

$$\begin{aligned} \mathcal{V} &= \{G^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0) - 0\}/\hat{\lambda} \\ &= \{G^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0) \cap D_r^{-1}(0) - 0\}/\hat{\hat{\lambda}}, \end{aligned} \tag{10}$$

where the S^1-action, $\hat{\hat{\lambda}}$, in the latter (symplectic) quotient is the $|\lambda| = 1$ restriction of the $\mathbb{C}^*$-action (1) in the former (holomorphic) quotient.

$\mathcal{V}^+$ ($\mathcal{V}^-$) shall denote the restriction of $\mathcal{V}$ to positive (negative) values of r in Eq. (5).

[1] To avoid obscuringly complicated notation, we focus on a simple example and discuss generalizations later.

Remark. As we show in more detail in Section 2, the r dependence in U_r, in Eq. (5), turns the 'ground state variety' into a 1-parameter family of (stratified) varieties[2], and the subtraction of zero in Eq. (10) separates the two branches, $\mathcal{V}^{\pm}$, defined with $r > 0$ and $r < 0$, respectively. Moreover, the dependence on r, as defined originally in the gauged linear σ-model [23], is complicated near $r = 0$ by quantum corrections and we restrict to $r \neq 0$.

Our main result is the following theorem.

Theorem 1. *Let $\mathcal{V}$, the 'ground state variety' (see Definition 1) of the gauged linear σ-model [23]. Suppose that the polynomial $G(s)$ is non-transversal at n isolated rays, $s_i^{\sharp}$, $i = 1, \ldots, n$. Then*

1. *$\mathcal{V}^+$ is a stratified variety [9] and $\mathcal{V}^+ = \mathcal{M}^{\sharp} \cup \bigcup_i A_i$ with $\mathcal{M}^{\sharp} = G^{-1}(0)$ is a projective hypersurface with n isolated nodes, where the n non-compact antennae components $A_i \simeq \mathbb{C}^1$ are attached.*
2. *When $\dim_{\mathbb{C}} \mathcal{V}^+ = 3$, the minimal holomorphic compactification $\bar{\mathcal{V}}^+ = \mathcal{M}^{\sharp} \cup \bigcup_i \bar{A}_i$ satisfies:*
   a. *$\bar{\mathcal{V}}^+$ is an exoflop of the small resolution(s) of $\mathcal{M}^{\sharp}$ in the sense of [2], and*
   b. *$\oplus_q H^{2q}(\bar{\mathcal{V}}^+)$ satisfies the "Kähler package" of requirements [9], and is compatible with mirror symmetry [24] and string theory [17,22].*

Remark (on generalizations). Generalizations of the above construction involve: (a) additional p and s variables, (b) additional corresponding terms in the superpotential (3): $W \to \sum_i p^i G_i(s)$, and (c) additional maps (1) and their modifications where the exponents of $\hat{\lambda}$ are different integers: negative for the p's, positive for the s's. The generalization (c) turns the $\mathbb{C}^*$ action (1) into a more general toric action, while (a) enlarges the field space and (b) modifies the "moment map" (3) accordingly. For a Calabi-Yau model, $\prod_i G_i(s)$ is a fixed to be section of the anticanonical bundle of the toric variety $(\mathcal{F}_s - \Delta)/\hat{\lambda}_s$, where $\hat{\lambda}_s$ is the restriction of $\hat{\lambda}$ to $\mathcal{F}_s \not\equiv \mathcal{F}|_{p^i=0}$ and Δ its fixed point set. The resulting 'ground state varieties' will thus include intersections of hypersurfaces in products of toric varieties [2,8,12,23]. All of these are of the form given in the Definition 1, with the 'ingredients' $\{(p, s), \hat{\lambda}, D_r, W\}$ duly modified. It should be clear that the main Theorem 1 then generalizes to the full class of intersections of hypersurfaces in products of toric varieties. Owing to Bertini's theorem, their "mild degenerations" studied herein form a subclass of codimension one, and connect the moduli spaces of all known Calabi-Yau three-folds [4,10,16,21].

Remark (on applications). Within the framework of Witten's σ-model [23], our analysis embeds these algebro-geometric results into string theory, and also gives an algebro-geometric interpretation of the string-theoretic results [17,22]. However, in this sense, the present results correspond only to the "untwisted sector" and only in "closed string theories".

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[2] The general category of 'stratified varieties' is specified for example in the works [9]. Our situation is far simpler: we will encounter unions of several (complex, algebraic) varieties of complex dimension $0, \ldots, 3$, possibly connected at codimension ≥ 1 subspaces.

In particular, the elements of the “twisted sector” (related to the non-trivial $\mathbb{C}^*$-action on the antennae) and the “fractional branes” (associated to the nodes) particular to “open string theories” [1,13–15,18,19] remain without a corresponding image at present. Nevertheless, in addition and in agreement with Ref. [2], our results prove that the codimension one subgeneric spacetime in string theory is a non-trivial stratified pseudovariety.

This article is organized as follows: Section 2 shows that the ‘ground state variety’ becomes stratified as $G(s)$ becomes non-transversal, and we explore the induced (exo-)strata[3] and their union. Section 3 explores the contribution of the induced (exo-)strata to the homology of the ‘ground state variety’. Section 4 re-examines the ‘conifold transition’ of Ref. [4] in view of Theorem 1.

2. The ground state variety

We now turn to analyze the geometry of the ground state variety, as determined by the choice of the homogeneous holomorphic polynomial $G(s)$. Such polynomials typically depend on a multitude of parameters; when properly accounted for redundancies, these span (a subspace of) the moduli space of the ground state variety. Thus, we automatically have a family of ground state varieties, fibered over this (partial) moduli space. Works in the literature, Ref. [23] and the subsequent studies, all assumed $G(s)$ to be transversal and so have explored the generic fibre of this family. We begin by analyzing this case in some detail, and then turn to the less generic mild degenerations of the fibre.

2.1. The transversal case

$G(s)$ being transversal[4], $G = \mathrm{d}G = 0$ only at $s = 0$. In this case, the zero locus of $\partial W = (G, p \cdot \partial_s G)$ is a union of two branches [5]:

$$(G)^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0) = \{p = 0, s : G(s) = 0\} \cup \{p, s = 0\}. \tag{11}$$

So, following the first (holomorphic quotient) part of Definition 1, we have that

$$\mathcal{V} = \{p = 0, s : G(s) = 0\}/\hat{\lambda} \cup \{p, s = 0\}/\hat{\lambda}, \tag{12}$$

where the quotients are taken after the fixed point of the $\hat{\lambda}$-action, $\{s, p = 0\}$, is excised. Now, since $\{s \neq 0\}/\hat{\lambda}$ is $\mathbb{P}^4$, then

$$\{p = 0, s \neq 0 : G(s) = 0\}/\hat{\lambda} = \mathcal{M} \tag{13}$$

is the Calabi-Yau quintic hypersurface in $\mathbb{P}^4$.

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[3] We will use the prefix ‘exo’ to denote (components of) strata that are ‘external’ to the ‘main’ stratum.

[4] Transversality ensures that the projective hypersurface defined by $G = 0$ is smooth.

[5] The subsequent analysis for non-transversal G, the case of our real interest, is more detailed and shown below. The Reader can then recover the presently omitted details as a special case; see also Ref. [23].

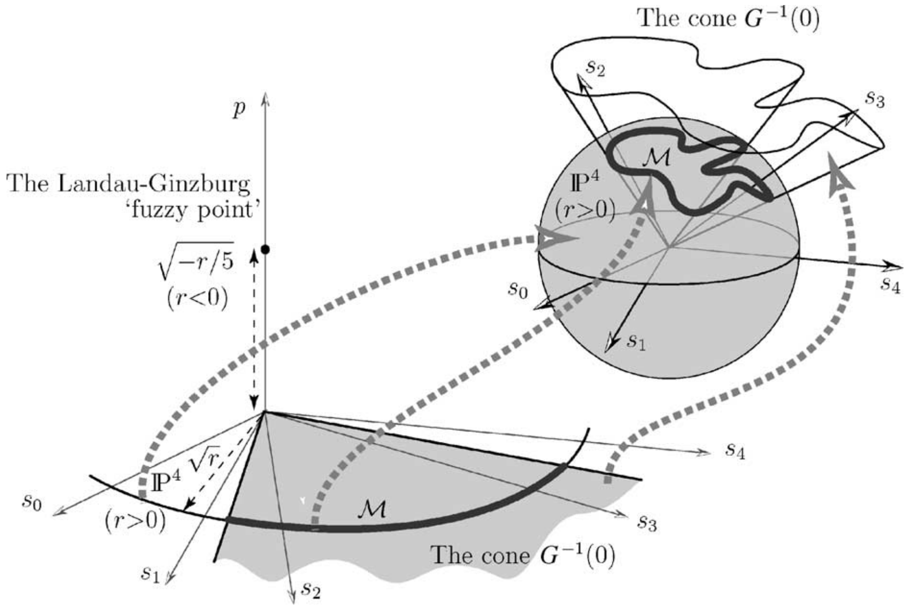


Fig. 1. The ground state variety, $\mathcal{V} \in \mathcal{F}$, and its 'geometric' phase, $\mathcal{M} \in \mathcal{F}|_{p=0}$, in the top right inset. This inset represents $\mathcal{M} = G^{-1}(0) \cap \mathbb{P}^4$, in $\mathcal{F}|_{p=0} \approx \mathbb{C}^5$ spanned by $s = (s_0, \ldots, s_4)$. The dashed grey arrows identify the image under the projection along p of this in the full field space, $\mathcal{F} \approx \mathbb{C}^6$, spanned by $(p, s_0, \ldots, s_5)$.

The second term in the union (12) is

$$\{p \neq 0, s = 0\}/\hat{\lambda} \simeq \mathbb{C}^*/\mathbb{C}^* \simeq \{\text{pt.}\}/\mathbb{Z}_5. \tag{14}$$

This is the 'fuzzy point' [2] of the Landau–Ginzburg orbifold. $\mathbb{Z}_5$ is the subgroup of $\hat{\lambda}$ which leaves $G(s)$ invariant and so acts trivially on both $W = p \cdot G(s)$ and on p.

The two quotients in the union (12) are thus manifestly disconnected: the former, (13), lies entirely in the $\{p = 0, s \neq 0\}$-subspace of the field space $\mathcal{F}$, whereas the latter, (14), lies well in the complementary $\{p \neq 0, s = 0\}$-subspace. The above is illustrated in Fig. 1. Even in the transversal case, $\mathcal{V}$ may be regarded as a stratified variety, consisting of two disconnected objects: a (complex) three-dimensional one and a (complex) zero-dimensional one, each of which containing a single variety: $\mathcal{M}$ and $\{\text{pt.}\}/\mathbb{Z}_5$, respectively.

In fact, the second component, $\{\text{pt.}\}/\mathbb{Z}_5$, actually lies in the 'second sheet' of the field space $\mathcal{F}$. To see this, it will be useful to also present $\mathcal{V}$ using the alternate (symplectic quotient) Definition 1:

1. When $r \gg 0$, $D_r^{-1}(0) \neq 0$ implies that $\|s\|^2 \neq 0$, and so $\partial_s G \neq 0$ as G is transversal. Then, $(\partial W)^{-1}(0)$ lies entirely in the $(p = 0)$ s-hyperplane, and $\mathcal{V}$ is the $\hat{\lambda}$-quotient, i.e., the complex base of the cone $G^{-1}(0)$. Then, $\mathcal{V} = \mathcal{M} \equiv [\{p = 0\} \cap G^{-1}(0)]/\hat{\lambda}$. Since $\{s \neq 0\}/\hat{\lambda} = \mathbb{P}^4$, the *projective* Calabi-Yau quintic hypersurface (13) is $G^{-1}(0)/\hat{\lambda} = \mathcal{M} \hookrightarrow \mathbb{P}^4$.

2. When $r \ll 0$, $D_r^{-1}(0) \neq 0$ implies that $|p|^2 \neq 0$, and so $\|s\|^2 = 0$ since G is transversal. Then, $(\partial W)^{-1}(0)$ lies entirely in complex the p-plane, and is the 'fuzzy point' [2], $\{|p| = \sqrt{|r|/5}\}/\mathbb{Z}_5$, of the Landau–Ginzburg orbifold (14).

At the critical point $r = 0$, these two branches formally collapse to the highly degenerate point $p, s = 0$, which is the branching point of the two 'sheets' of the field space $\mathcal{F}$. This point is, by definition, excised before taking the quotients (2). Indeed, for applications to string theory, the preceding analysis is not to be trusted in the region near $p, s = 0$ since quantum corrections modify the map (5) and so also the structure of the quotients in Definition 1; see Ref. [23]. For this reason, we will mostly concern ourselves with the $r \gg 0$ 'sheet', and comment on occasion on the $r \ll 0$ 'sheet', but leave any 'connection' between the two 'sheets' unexplored for now.

2.2. The conifold with exocurves

Unlike Ref. [23] and subsequent work, we will be concerned with ground state varieties using homogeneous holomorphic polynomials $G(s)$ which are non-transversal along n isolated (complex) directions:

$$\partial G(s) = 0 \Rightarrow s = s_j^\sharp, \quad j = 1, \ldots, n. \tag{15}$$

Clearly, $\{s_j^\sharp\} \simeq \mathbb{C}^1$, and we denote by $\mathcal{B}^n \stackrel{\text{def}}{=} \sqcup_{j=1}^n \{s_j^\sharp\}$ the 'bouquet' of n $\mathbb{C}^1$'s all meeting at the origin. Since $G(s)$ is holomorphic and homogeneous, $\partial G(s) = 0$ implies $s \cdot \partial_s G(s) = 5G(s) = 0$ and the $G(s) = 0$ condition is automatically satisfied on $\mathcal{B}^n$. Thus, we find that

$$(G)^{-1}(0) \cap (p \cdot \partial_s G)^{-1}(0) = \{p = 0, s \neq s_j^\sharp : G(s) = 0\} \cup \{\{p\} \times \mathcal{B}^n\}. \tag{16}$$

So, following the first (holomorphic quotient) part of Definition 1, we now have that

$$\mathcal{V} = \{p = 0, s \neq s_j^\sharp : G(s) = 0\}/\hat{\lambda} \cup \{\{p\} \times \mathcal{B}^n\}/\hat{\lambda}; \tag{17}$$

again, the quotients are taken after the fixed point of the $\hat{\lambda}$-action, $\{s, p = 0\}$, is excised. Now, since $\{s \neq 0, s_j^\sharp\}/\hat{\lambda}$ equals $\mathbb{P}^4$ without its points where $G(s)|_{\mathbb{P}^4}$ is non-transversal, then

$$\{p = 0, s \neq 0, s_j^\sharp : G(s) = 0\}/\hat{\lambda} = \mathcal{M}^\sharp - \mathrm{Sing}(\mathcal{M}^\sharp) \tag{18}$$

is the non-singular (and non-compact) part of the conifold[6] $\mathcal{M}^\sharp \hookrightarrow \mathbb{P}^4$. Note that

$$\begin{aligned}\mathrm{Sing}(\mathcal{M}^\sharp) &\stackrel{\text{def}}{=} \{p = 0, s = s_j^\sharp : G(s) = 0\}/\hat{\lambda} \\ &= \bigcup_{j=1}^n \{p = 0, s = s_j^\sharp\}/\hat{\lambda} = \bigcup_{j=1}^n x_j^\sharp \subset \mathcal{M}^\sharp,\end{aligned} \tag{19}$$

[6] Following Ref. [4], a *conifold* is a variety which is smooth except for a finite number of isolated conical singularities. Furthermore, herein we will consider only varieties with nodes (double points).

since $G(s_j^\sharp) = 0$; $x_j^\sharp$ are the singular points of

$$\mathcal{M}^\sharp \overset{\text{def}}{=} [\{p = 0\} \cap G^{-1}(0)]/\hat{\lambda}. \tag{20}$$

The second quotient in the union (17) is quite more intricate. Setting $p = 0$ in Eq. (5), we see that $\{\{p\} \times \mathcal{B}^n\}/\hat{\lambda}$ is non-empty in the $r > 0$ 'sheet' of the field space $\mathcal{F}$, and also that it includes the points $\{p = 0, s_j^\sharp \neq 0\}/\hat{\lambda} = \mathrm{Sing}(\mathcal{M}^\sharp)$. On the other hand,

$$(\{\{p\} \times \mathcal{B}^n\}/\hat{\lambda}) = \sqcup_{j=1}^n A_j, \quad A_j \overset{\text{def}}{=} \{p, s_j^\sharp\}/\hat{\lambda}. \tag{21}$$

Each of the A_j's contains precisely one of the singular points of $\mathcal{M}^\sharp$, as given in Eq. (19)

$$x_j^\sharp = \{p = 0, s_j^\sharp\}/\hat{\lambda} = A_j \cap \mathcal{M}^\sharp. \tag{22}$$

Thus, ground state variety (17), which is the *connected* union of (21) and of (18), is then

$$\mathcal{V} = \mathcal{M}^\sharp \cup \bigsqcup_{j=1}^{n} A_j. \tag{23}$$

That is, the ($r > 0$ 'sheet' of the) ground state variety is the conifold $\mathcal{M}^\sharp$, with an exocurve, A_j, attached at each singular point.

In the other, $r < 0$ 'sheet' of the field space $\mathcal{F}$, the first term in the union (5) turns out to be empty. On the other hand, the second one is not since $\{p, s = s_j^\sharp\}$ does include the complex p-plane in which Eq. (5) shows that $r < 0$. In this case, the second term in the unio (17) again turns out to be of the form (21), except this time the A_j's have a single common point, the Landau–Ginzburg orbifold (14).

Alternatively, consider the symplectic quotient: impose the vanishing of D_r, i.e., intersect with $D_r^{-1}(0)$, and pass to the S^1-quotient. To this end, consider each term in the union (16) separately.

- $r > 0$

  Now $D_r = 0$ implies that

$$\|s\|^2 - 5|p|^2 = r > 0, \Rightarrow \|s\|^2 \geq |r| > 0. \tag{24}$$

  The ground state variety now is the S^1-quotient of the union:

$$\{p = 0, s \neq s_j^\sharp : G(s) = 0, \|s\|^2 = r\} \cup \{p, s = s_j^\sharp : \|s\|^2 = r_+\}, \tag{25}$$

  where

$$r_+ = 5|p|^2 + |r|. \tag{26}$$

  Note that the $p = 0$ points of the second component, where $\|s\|^2 = r_+ = r$, the

$$\{(p, s) = (0, s_j^\sharp) : \|s\|^2 = r\} \tag{27}$$

  points are the $s \to s_j^\sharp$ limiting points of the first component, since $G(s_j^\sharp) = 0$. The S^1 quotient of these are the (nodal) singular points of the conifold (20), and they connect the two terms in the union (25). This then becomes $\mathcal{M}^\sharp \cup \sqcup_j A_j$, just as obtained using the holomorphic quotient (23).

- $r < 0$ Now $D_r = 0$ implies that

$$\|s\|^2 - 5|p|^2 = r < 0, \Rightarrow |p| \geq \sqrt{|r|/5} > 0. \tag{28}$$

This renders the first term in the union (12) empty, and the ground state variety now is:

$$\{p, s = s_j^\sharp : \|s\|^2 = r_-\}/S^1, \tag{29}$$

where now

$$r_- = \|s_j^\sharp\|^2 + |r|. \tag{30}$$

Note that at $s = 0 = r_-$, where $|p| = \sqrt{|r|/5}$, the

$$\{p, s = 0 : 5|p|^2 = r\}/S^1 \tag{31}$$

point is common to all components of the second component, and is the 'fuzzy point' of the Landau–Ginzburg orbifold (14).

The foregoing proves the following lemma.

Lemma 1. *With the* '*ingredients*', $\{(p, s), \hat{\lambda}, D_r, W\}$, *defined as in* Eqs. (1) and (5) *and* (3), *the* $r > 0$ '*sheet*' *of the ground state variety* (Definition 1), $\mathcal{V}^+$, *becomes a stratified variety*, $\mathcal{M}^\sharp \cup \sqcup_j A_j$ *when* $G(s)$ *is non-transversal as specified in* Eq. (15).

Remark. The 'main' stratum (18) has complex dimension 3, while the 'exocurves' (21) minus the singular points $x_j^\sharp$ form the complex dimension 1 stratum; the singular points, $\sqcup_j x_j^\sharp = \mathrm{Sing}(\mathcal{M}^\sharp)$, form the complex dimension 0 stratum.

Corollary 1. *Under the same conditions as in* Lemma 1, *the* $r < 0$ '*sheet*' *of the ground state variety*, $\mathcal{V}^-$, *is the stratified variety*: *the union of the exocurves* (21), $\cup_j A_j$, *connected at the* '*fuzzy point*' *of the Landau–Ginzburg orbifold* (14).

Remark. The $r < 0$ stratified variety consists of the exocurves A_j minus the 'fuzzy point' which form the complex dimension 1 stratum, and the 'fuzzy point' (14) which forms the complex dimension 0 stratum.

The resulting non-transversal ground state variety is illustrated in Fig. 2.

Remark. Since the 'fuzzy point' of the Landau–Ginzburg orbifold (14) may be, formally, considered as the (negative size) collapse (or, perhaps more properly, analytic continuation) of the three-dimensional Calabi-Yau variety $\mathcal{M}$, the same relation remains between $\mathcal{V}^+$ and $\mathcal{V}^-$, regardless of the (non)transversality of $G(s)$.

2.3. *The exocurves*

We now turn to study the exocurves, A_j, in some detail. In particular, we prove the following lemma.

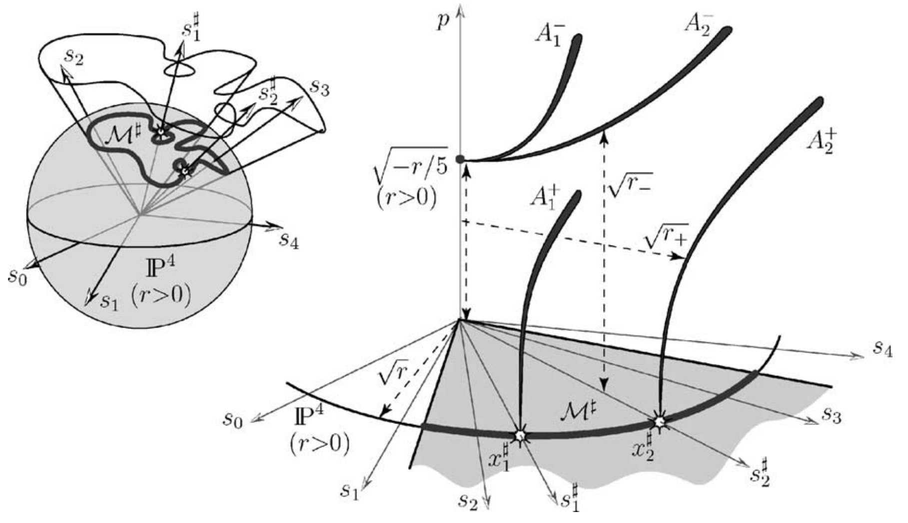


Fig. 2. A non-transversal ground state variety, $\mathcal{V} \in \mathcal{F}$, and its 'geometric' phase, $\mathcal{M}^\sharp = G^{-1}(0) \cap \mathbb{P}^4$, in the top left inset. The rays $s_j^\sharp$ pass through the nodes of $\mathcal{M}^\sharp$, $x_j^\sharp$, which is where the exocurves, A_j^+, attach to $\mathcal{M}^\sharp$ in the $r > 0$ 'sheet'. In the $r < 0$ 'sheet', the exocurves A_j^- all meet at the Landau–Ginzburg 'fuzzy point'.

Lemma 2. *In the $r > 0$ 'sheet' of the field space, $\mathcal{F}$, the exocurves (21) are*

$$A_j^+ \simeq \mathbb{CP}^1_{[-5,1]} \simeq \mathbb{C}^1. \tag{32}$$

Proof. In the $r > 0$ 'sheet', the definition (21) of the exocurve:

$$A_j^+ \stackrel{\text{def}}{=} \{p, s_j^\sharp\}/\hat{\lambda}, \tag{33}$$

includes implicitly that $\|s\|^2 \geq |r| > 0$ owing to Eq. (24), and the superscript '+' reminds that $r > 0$. That is,

$$(p, s_j^\sharp) \cong (\lambda^{-5} p, \lambda s_j^\sharp), \quad \lambda \in \mathbb{C}^*, \tag{34}$$

which defines A_j^+ as the weighted projective space $A_j^+ = \mathbb{P}^1_{[-5,1]}$, proving the first part of (33). This case, however, differs from the usual consideration of weighted projective spaces [5] in that the weights, -5 and 1, are of opposite sign. Still, we proceed by considering the two candidate charts:

$$U_p = (p, s_j^\sharp)_p \cong (1, u_p), \quad p \neq 0, \;\; u_p \stackrel{\text{def}}{=} s_j^\sharp\, p^{1/5} \tag{35}$$

and

$$U_s = (p, s_j^\sharp)_s \cong (u_s, 1), \quad s_j^\sharp \neq 0, \;\; u_s \stackrel{\text{def}}{=} p (s_j^\sharp)^5. \tag{36}$$

In both cases, the equivalences are obtained using the map (34), however with $\lambda = p^{1/5}$ in the first case, and $\lambda = (s_j^\sharp)^{-1}$ in the second. Now, in the second candidate chart, U_s, the limit point $p, u_s \to 0$ is included, and so

$$U_s = (p, s_j^\sharp)_s \cong (u_s, 1) \simeq \mathbb{C}^1 \tag{37}$$

is a proper chart. On the other hand, in the first candidate chart, U_p, the limit point $s_j^\sharp$, $u_p \to 0$ is *excluded* by the inequality (24), so that

$$U_p = (p, s_j^\sharp)_p \cong (1, u_p) \simeq \mathbb{C}^* \tag{38}$$

is *not* a proper chart. In its place, we should introduce two $\mathbb{C}^1$-like charts which cover U_p. However, this will not really be necessary since Eqs. (35) and (36) imply that

$$u_s = p(s_j^\sharp)^5 \mapsto u_p^5, \tag{39}$$

which is a 1-to-5 holomorphic map outside $u_s = 0$. That is, $U_p = (1, u_p) \simeq \mathbb{C}^*$ is a five-fold cover of $(U_s - 0) = (u_s, 1)_{u_s \neq 0} \simeq \mathbb{C}^*$; $u_s = 0$ is of course the branching point of this holomorphic covering. Therefore, A_j^+ may be parametrized by u_s and so $A_j^+ \simeq U_s$. With (39) as the 'glueing map', we then have that the jth exocurve is:

$$A_j^+ \simeq \mathbb{P}^1_{[-5,1]} = U_p \cup U_s = U_s \simeq \mathbb{C}^1. \tag{40}$$

□

Lemma 3. *In the* $r < 0$ *'sheet' of the field space*, $\mathcal{F}$, *the exocurves* (21) *are*

$$A_j^- \simeq \mathbb{CP}^{1,-}_{[-5,1]} \simeq \mathbb{C}^1/\mathbb{Z}_5. \tag{41}$$

Proof. In the $r < 0$ 'sheet', the definitions (33) and (34) still guarantees that $A_j^- \simeq \mathbb{P}^{1,-}_{[-5,1]}$, but now Eq. (28) enforces $|p|^2 \geq |r| > 0$, as indicated by the superscript '$-$'. We again proceed by considering the two candidate charts (35) and (36). This time, it is in the first candidate chart, U_p, that the limit point $s_j^\sharp$, $u_p \to 0$ is included, and so

$$U_p = (p, s_j^\sharp)_p \cong (1, u_p) \simeq \mathbb{C}^1 \tag{42}$$

is a proper chart. Similarly, in is now the second candidate chart, U_s, from which the limit point $p, u_s \to 0$ is *excluded* by the inequality (28), so that

$$U_s = (p, s_j^\sharp)_s \cong (u_s, 1) \simeq \mathbb{C}^* \tag{43}$$

is *not* a proper chart. Again, it is not necessary to introduce two $\mathbb{C}^1$-like charts to cover U_s, since Eqs. (35) and (36) again imply the 1-to-5 holomorphic map (39) now outside $u_p = 0$. Now $(U_p - 0) = (1, u_p \neq 0) \simeq \mathbb{C}^*$ is a five-fold cover of $U_s = (u_s, 1) \simeq \mathbb{C}^*$, and $u_p = 0$ is of course the branching point of this holomorphic covering. Therefore, A_j^- now must be parametrized by u_p which is five-fold redundant except at $u_p = 0$. Therefore, we now have that the jth exocurve is:

$$A_j^- \simeq \mathbb{P}^{1,-}_{[-5,1]} = (U_p \cup U_s)/\mathbb{Z}_5 = U_p/\mathbb{Z}_5 \simeq \mathbb{C}^1/\mathbb{Z}_5. \tag{44}$$

□

Remark. Note that the $\mathbb{Z}_5$ quotient in Lemma 3 precisely corresponds to the $\mathbb{Z}_5$ quotient in Eq. (14). Indeed, this says that the "fuzzy point" of the Landau–Ginzburg orbifold (14) becomes

$$(\sqcup_{j=1}^n A_j^-) \simeq (\mathbb{C}^1/\mathbb{Z}_5)^{\vee 5}, \tag{45}$$

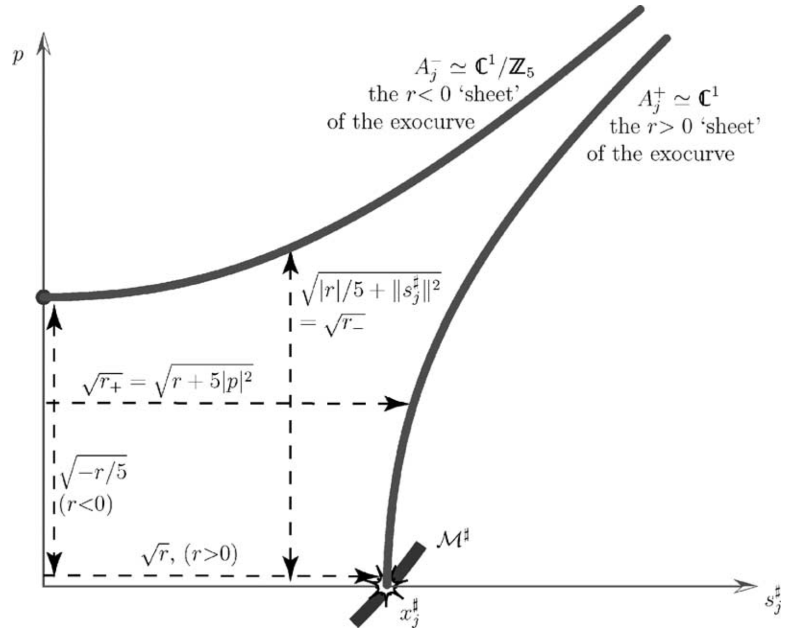


Fig. 3. Over the non-transversal rays, $s_j^\sharp$, of $G(s)$ the complex variable p is subject only to the projectivization action, $\hat{\lambda}$. The resulting space, the exocurve $\mathbb{P}^1_{[-5,1]} = \{p, s_j^\sharp\}/\hat{\lambda}$, is illustrated here for both the $r > 0$ 'sheet' (A_j^+), and the $r < 0$ 'sheet' (A_j^-).

i.e., the 'plum product' of five copies of the $\mathbb{C}^1/\mathbb{Z}_5$ cone, all connected at the vertex—the "fuzzy point" (14).

The exocurves are illustrated in Fig. 3.

2.4. A comparison

For comparison, we include a similar analysis of $\mathbb{P}^1_{[5,1]}$. In contrast to the non-compact $\mathbb{P}^1_{[-5,1]}$, the weighted projective space $\mathbb{P}^1_{[5,1]}$ will prove to be compact.

Again, it is possible to view $\mathbb{P}^1_{[5,1]}$ both as a holomorphic quotient,

$$\mathbb{P}^1_{[5,1]} \simeq \{q, s\}/\hat{\mu} \tag{46}$$

where

$$\hat{\mu} : (q, s) \mapsto (\mu^5 q, \mu s), \quad \mu \in \mathbb{C}^* \tag{47}$$

and also as a symplectic quotient,

$$\mathbb{P}^1_{[5,1]} \simeq \{\{q, s\} \cap \Delta_r^{-1}(0)\}/S^1 \tag{48}$$

where the S^1-action is the restriction of (47) to $|\mu| = 1$ and Eq. (5) now becomes

$$\Delta_r = |s|^2 + 5|q|^2 - r, \quad r \in \mathbb{R}. \tag{49}$$

Following the proofs of Lemmas 2 and 3, we consider the latter.

The vanishing of Δ_r now simply states that

$$|s|^2 + 5|q|^2 = r \geq 0, \tag{50}$$

where $r = 0$ would force $s = 0 = q$, the trivial solution. Restricting then to $r \neq 0$, Eq. (50) implies the positive definiteness of r on $\Delta_r^{-1}(0)$, so that there is only the $r > 0$ sheet. Indeed, this is precisely why the $r < 0$ sheet of $\mathbb{P}^1_{[-5,1]}$ appears to be rather unfamiliar an object. Owing to the inequality (50), s, q must not vanish *simultaneously*; either one of them however may very well vanish while the other one is non-zero. Thus, unlike in the case of $\mathbb{P}^1_{[-5,1]}$, we now have two perfectly proper coordinate charts:

$$U_q = (q, s)_q \cong (1, u_q) \simeq \mathbb{C}^1, \quad \text{using } \mu = q^{-1/5}, \ q \neq 0, \ u_q = sq^{-1/5} \tag{51}$$

and

$$U_s = (q, s)_s \cong (u_s, 1) \simeq \mathbb{C}^1 \quad \text{using } \mu = s^{-1}, \ s \neq 0, \ u_s = qs^{-5}. \tag{52}$$

The two chart coordinates, u_q and u_s, respectively, *can* attain the value of 0, since $s, u_q \to 0$ is permitted in U_q where $q \neq 0$, and $q, u_s \to 0$ is permitted in U_s where $s \neq 0$. Finally, the two charts are glued through the relation

$$u_s = u_q^{-5}, \quad \text{where } u_s, u_q \neq 0, \tag{53}$$

which provides a 1–5 map:

$$\{U_s - 0\} \overset{1\text{–}5}{\to} \{U_q - 0\}. \tag{54}$$

To render the map (53) single-valued, we may glue together U_s and $U_q/\mathbb{Z}_5 : 0 \in U_s$ becomes '∞' added to $U_q/\mathbb{Z}_5$, and $0 \in U_q/\mathbb{Z}_5$ becomes '∞' added to U_s. The resulting space, $\mathbb{P}^1_{[5,1]} = U_s \cup (U_q/\mathbb{Z}_5)$, then is compact and smooth except at $0 \in U_q/\mathbb{Z}_5$, where $\mathbb{P}^1_{[5,1]}$ has a $\mathbb{Z}_5$ quotient singularity, i.e., a conical singularity with $(1 - (1/5))2\pi = 8\pi/5$ deficit angle.

Note, however, that by Delorme's lemma [6], $\mathbb{P}^1_{[k,1]} \approx \mathbb{P}^1_{[1,1]} \equiv \mathbb{P}^1 \simeq S^2$. The relationship '$\approx$' here denotes a k-to-1 map of the coordinates as used here, but an isomorphism of the corresponding coordinate rings, which then extends to an isomorphism of the respective spaces [7].

2.5. *A one-point compactification of exocurves*

For later convenience and use, we describe here a one-point compactifications of the exocurves.

It is straightforward from the proof of lemmas (2) that the limiting point $s, u_p \to \infty$ may be added to U_p. Upon the inversion of its variables, this now becomes a proper coordinate chart:

$$\tilde{U}_p = (1, w_p) \simeq \mathbb{C}^1, \quad w_p \overset{\text{def}}{=} u_p^{-1} = s_j^{\sharp-1} p^{-1/5}. \tag{55}$$

Clearly, the glueing map now becomes

$$u_s = w_p^{-5} : \{U_s - 0\} \stackrel{1\text{–}5}{\longrightarrow} \{\tilde{U}_p - 0\}. \tag{56}$$

To render the glueing map (56) single-valued, we form

$$\bar{A}_j^+ \stackrel{\text{def}}{=} U_s \cup (\tilde{U}_p/\mathbb{Z}_5) \simeq \mathbb{P}^1_{[5,1]}. \tag{57}$$

As shown above, this is compact and isomorphic to $\mathbb{P}^1 \simeq S^2$.

3. Cohomology and homology of $\bar{\mathcal{V}}$

As presented in Theorem 1, the stratified variety can be written as

$$\bar{\mathcal{V}} = \mathcal{M}^\sharp \cup \bigcup_{j=1}^{n} \bar{A}_j, \quad \mathcal{M}^\sharp \cap \bar{A}_j = x_j^\sharp. \tag{58}$$

The Mayer–Vietoris principle then induces the long exact cohomology sequence

$$\begin{aligned} \cdots \to H^q(\bar{\mathcal{V}}) &\to H^q(\mathcal{M}^\sharp) \oplus H^q(\cup_j \bar{A}_j) \to H^q(\mathcal{M}^\sharp \cap \cup_j \bar{A}_j) \\ &\to H^{q+1}(\bar{\mathcal{V}}) \to \cdots \end{aligned} \tag{59}$$

Since

$$\mathcal{M}^\sharp \cap \cup_j \bar{A}_j = \cup_j(\mathcal{M}^\sharp \cap \bar{A}_j) = \sqcup_j x_j^\sharp, \tag{60}$$

owing to our assumption that $x_j^\sharp$ are isolated (non-overlapping) nodes, and

$$H^q(\cup_j x_j^\sharp) = \bigoplus_{j=1}^{n} H^q(x_j^\sharp) \simeq \delta_{q,0}\mathbb{C}^{\oplus n}, \tag{61}$$

the long exact sequence (59) breaks into five isomorphisms:

$$H^q(\bar{\mathcal{V}}) = H^q(\mathcal{M}^\sharp) \oplus H^q(\cup_j \bar{A}_j), \quad \text{for } q = 2, \ldots, 6 \tag{62}$$

and

$$0 \to H^0(\bar{\mathcal{V}}) \xrightarrow{\alpha} H^0(\mathcal{M}^\sharp) \oplus H^0(\cup_j \bar{A}_j) \xrightarrow{\beta} H^0(\sqcup_j x_j^\sharp) \to H^1(\bar{\mathcal{V}}) \to 0. \tag{63}$$

The above map β is induced from the injective inclusion $\sqcup_j x_j^\sharp = \mathcal{M}^\sharp \cap (\sup_j \bar{A}_j) \to \mathcal{M}^\sharp \sqcup (\cup_j \bar{A}_j)$, and so is surjective. Then:

$$H^1(\bar{\mathcal{V}}) = \emptyset \quad \text{and} \quad H^0(\bar{\mathcal{V}}) \simeq \mathbb{C}. \tag{64}$$

3.1. Contributions from the antennae

Recall from a previous section that $\bar{A}_j \simeq \mathbb{P}^1 \simeq S^2$. Then,

$$H^q(\bar{A}_j) \begin{cases} \simeq \mathbb{C}^1 & \text{for } q = 0, 2, \\ = \emptyset & \text{otherwise.} \end{cases} \tag{65}$$

With this, for $\mathcal{M}^\sharp$ with n simple nodes, Eqs. (62) and (64) would seem to imply that $H^q(\mathcal{V})$ should equal to $H^q(\mathcal{M}^\sharp)$, except for $q = 2$, where it ought to be augmented by $H^2(\cup_j \bar{A}_j) \simeq \oplus\mathbb{C}^{\oplus n}$.

This, however, is not correct: the (area) 2-forms of the n antennae are not independent cohomology elements. As described in detail in Ref. [16], N mutually exclusive subsets of the $nx_j^\sharp$'s lie on corresponding 4-cycles $C_k^{(4)} \subset \mathcal{M}^\sharp$, $k = 1, \ldots, N$. Let J_k denote the multi-index containing the indices, j, of all $x_j^\sharp$'s that lie on $C_k^{(4)}$. Clearly then,

$$\bar{A}_j \cap C_k^{(4)} = \begin{cases} x_j^\sharp & \text{if } j \in J_k \text{ i.e. } x_j^\sharp \in C_k^{(4)}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{66}$$

Considering then the *homology* elements in $H_q(\bar{\mathcal{V}})$, dual to the cohomology group obtained in Eqs. (62) and (64), and denoting them by square brackets, we have:

$$[\bar{A}_j] \cap [C_k^{(4)}] = \begin{cases} 1 & \text{if } j \in J_k, \\ 0 & \text{otherwise.} \end{cases} \tag{67}$$

Owing to this result, it follows that

$$[\bar{A}_j] = [\bar{A}_{j'}] \quad \text{if } j, j' \in J_k, \qquad [\bar{A}_j] \neq [\bar{A}_{j'}] \quad \text{otherwise.} \tag{68}$$

That is, the n antennae, $\{\bar{A}_j\}$ contribute only N inequivalent 2-cycles, so

$$H_2(\cup_j \bar{A}_j) \simeq \mathbb{C}^{\oplus N} \simeq H^2(\cup_j \bar{A}_j). \tag{69}$$

3.2. The combined result

Combining Eqs. (62), (64) and (69) proves the following lemma.

Lemma 4. *Let $\bar{\mathcal{V}}$ as defined in Eq. (58), where $\mathcal{M}^\sharp$ is a conifold with only n isolated nodes ($x_j^\sharp$) lying on N distinct 4-cycles $C_k^{(4)}$, and $\bar{A}_j$ as defined in Eq. (57). Then,*

$$H^q(\bar{\mathcal{V}}) = \begin{cases} H^q(\mathcal{M}^\sharp) & \text{for } q \neq 2, \\ H^2(\mathcal{M}^\sharp) \oplus H^2(\cup_j \bar{A}_j) \simeq H^2(\mathcal{M}^\sharp) \oplus \mathbb{C}^{\oplus N}. \end{cases} \tag{70}$$

As it stands, with $H^3(\bar{\mathcal{V}}) = H^3(\mathcal{M}^\sharp)$, the *complete* $H^*(\bar{\mathcal{V}})$ can have neither Poincaré duality nor a Hodge decomposition. Both are obstructed by the fact that the 3-cycle(s) which pass through the $x_j^\sharp$'s remain without dual 3-cycle(s) [16]. In fact, the subgroup of $H^3(\bar{\mathcal{V}})$

generated by the 3-cycles passing through the $x_j^\sharp$'s may well be odd-dimensional, making this obstruction manifest.

However, $\oplus_q H^{2q}(\bar{\mathcal{V}})$ subgroup does exhibit both Poincaré duality and a Hodge decomposition. As usual, $H^2(\bar{A}_j) \simeq \mathbb{C}^1$ is generated by the volume (1,1)-form on $\bar{A}_j \simeq \mathbb{P}^1$. Moreover, dually to the homology result (67), the volume (1,1)-forms, $\omega^j_{(1,1)}$, of all $\bar{A}_j$'s which intersect $C_k^{(4)}$ are dual to the (2,2)-form $\omega^k_{(2,2)}$, itself dual to $C_k^{(4)}$. In fact, the double dualities[7]

$$\omega^j_{(1,1)} \overset{*}{\sim} [\bar{A}_j] \overset{*}{\sim} [C_k^{(4)}] \quad \text{and} \quad [\bar{A}_j] \overset{\#}{\sim} [C_k^{(4)}] \overset{\#}{\sim} \omega^k_{(2,2)}, \quad \text{for } j \in J_k, \tag{71}$$

establishes the isomorphisms

$$\omega^j_{(1,1)} \simeq [C_k^{(4)}] \quad \text{and} \quad [\bar{A}_j] \simeq \omega^k_{(2,2)}, \quad \text{for } j \in J_k, \tag{72}$$

whereupon Eq. (68) implies that also

$$[\omega^j_{(1,1)}] = [\omega^{j'}_{(1,1)}] \quad \text{if } j, j' \in J_k, \qquad [\omega^j_{(1,1)}] \neq [\omega^{j'}_{(1,1)}] \quad \text{otherwise.} \tag{73}$$

On a more formal level, Eq. (67) implies that also

$$[\omega^j_{(1,1)}] \cup [\omega^k_{(2,2)}] = \begin{cases} 1 & \text{if } j \in J_k, \\ 0 & \text{otherwise.} \end{cases} \tag{74}$$

Clearly, the evaluation map of the cup product here cannot be the integration (of the wedge product of the indicated forms) over the stratified variety $\bar{\mathcal{V}}$ in any conventional sense. Instead, it may be taken to reduce to the evaluation over the point of common support, $x_j^\sharp = \bar{A}_j \cap C_k^{(4)}$ if $j \in J_k$, and is vacuous otherwise.

Owing to the isomorphisms (72), the $[\bar{A}_j] \sim^* [C_k^{(4)}]$ duality (when $j \in J_k$) implies the desired Poincaré duality of $\omega^j_{(1,1)} \sim^* \omega^k_{(2,2)}$, for all $j \in J_k$. Let $\langle \omega^k_{(2,2)} \rangle$ denote the subgroup of $H^{(2,2)}(\mathcal{M}^\sharp)$ generated by the $\omega^k_{(2,2)}$'s. The quotient $H^{(2,2)}(\mathcal{M}^\sharp)/\omega^k_{(2,2)}$ is then generated by the (2,2)-forms dual to 4-cycles which do not pass through $x_j^\sharp$; this quotient is easily seen to form an additive group, exhibiting both Poincaré duality and Hodge decomposition.

The foregoing then proves the following lemma.

Lemma 5. *Let $\bar{\mathcal{V}}$ as defined in* Eq. (58), *where $\mathcal{M}^\sharp$ is a conifold with only n isolated nodes ($x_j^\sharp$) lying on N distinct 4-cycles $C_k^{(4)}$, and $\bar{A}_j$ as defined in* Eq. (57). *Then,*

$$\oplus_q H^{2q}(\bar{\mathcal{V}}) = \begin{cases} H^{2q}(\mathcal{M}^\sharp) & \text{for } q \neq 1, \\ H^2(\mathcal{M}^\sharp) \oplus H^2(\cup_j \bar{A}_j) \simeq H^2(\mathcal{M}^\sharp) \oplus \mathbb{C}^{\oplus N}, \end{cases} \tag{75}$$

has both an induced Hodge decomposition and Poincaré duality, as induced by the double dualities (71).

[7] By $\sim^*$ we denote the standard homology–cohomology duality, and use $\sim^\#$ for Poincaré duality.

4. Deformations, resolutions and the mirror map

We have originally restricted $\mathcal{M}^\sharp$ to conifolds with only nodes (i.e., double points, or A_1 hypersurface singularities), $x^\sharp_j$. Their local neighborhood is isomorphic to the cone $\mathbb{C}^4/Q$, where Q is a non-degenerate quadratic polynomial over $\mathbb{C}^4$. In a small resolution, this neighborhood is replaced with a copy of the total space of an $\mathcal{O}(-1,-1)\stackrel{\text{def}}{=}\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ bundle over $\mathbb{P}^1 \simeq S^2$. In short, a small resolution replaces each node $x^\sharp_j$ with a $(-1,-1)$-curve, $\mathbb{P}^1_{\natural,j} \simeq S^2$. Since there are two topologically distinct ways to do this at each node $x^\sharp_j$, a conifold $\mathcal{M}^\sharp$ with n would appear to have 2^n small resolutions, $\mathcal{M}^\natural_I$. However, all nodes $x^\sharp_j$ which lie on a single 4-cycle $C^{(4)}_k \subset \mathcal{M}^\sharp$ must be resolved "compatibly": all the corresponding 2-spheres $\mathbb{P}^1_{\natural,j} \in \mathcal{M}^\natural_I$ intersect $C^{(4)}_{\natural,k} \in \mathcal{M}^\natural_I$ (the proper transform of $C^{(4)}_k \subset \mathcal{M}^\sharp$) in a single point and so must all represent the same element of $H_2(\mathcal{M}^\natural_I)$, the one that is dual to $C^{(4)}_{\natural,k}$. With the use of Eq. (66), this implies that

$$[\omega^{\natural,j}_{(1,1)}] = [\omega^{\natural,j'}_{(1,1)}] \quad \text{if } j, j' \in J_k, \qquad [\omega^{\natural,j}_{(1,1)}] \neq [\omega^{\natural,j'}_{(1,1)}] \quad \text{otherwise} \tag{76}$$

and

$$[\omega^{\natural,j}_{(1,1)}] \cup [\omega^{\natural,k}_{(2,2)}] = \begin{cases} 1 & \text{if } j \in J_k, \\ 0 & \text{otherwise,} \end{cases} \tag{77}$$

for

$$H_2(\mathcal{M}^\natural_I) \ni \mathbb{P}^1_{\natural,j} \stackrel{*}{\sim} \omega^{\natural,j}_{(1,1)} \in H^{(1,1)}(\mathcal{M}^\natural_I), \tag{78}$$

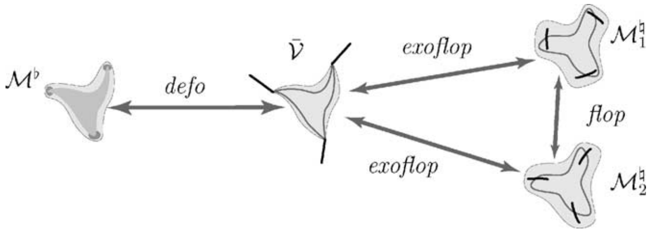
$$H_4(\mathcal{M}^\natural_I) \ni C^{(4)}_{\natural,k} \stackrel{*}{\sim} \omega^{\natural,k}_{(2,2)} \in H^{(2,2)}(\mathcal{M}^\natural_I). \tag{79}$$

Of course, in Eq. (77), the cup product is indeed obtained as the ordinary wedge product, integrated over the (smooth) manifold $\mathcal{M}^\natural_I$ Consequently, the multiplicity of small resolutions to $I = 1, \ldots, 2^N$, where N is the number of $H_2(\mathcal{M}^\natural_I)$ elements which the small resolution exceptional sets, $\mathbb{P}^1_{\natural,j}$ represent, i.e., the number of $H_4(\mathcal{M}^\natural_I)$ elements, $C^{(4)}_{\natural,k}$, which are the proper transforms of the 4-cycles that pass through the nodes $x^\sharp_j \in \mathcal{M}^\sharp$ [16].

The formal identity of the Eqs. (73) and (74) with the Eqs. (76) and (77) then proves the following lemma.

Lemma 6. *Let $\mathcal{M}^\sharp$ be a Calabi-Yau complex* 3*-dimensional algebraic variety with only a finite number of isolated nodes, $x^\sharp_j$. Let $\mathcal{M}^\natural_1$ and $\mathcal{M}^\natural_2$ denote two small resolutions of $\mathcal{M}^\sharp$, related by a flop: $\mathcal{M}^\natural_1 \stackrel{\text{f}}{\leftrightarrow} \mathcal{M}^\natural_2$. Finally, let $\bar{\mathcal{V}}$ be the compactification of the stratified variety*

(58). *Then the flop involution* $\mathcal{M}_1^\natural \leftrightarrow^{\mathrm{f}} \mathcal{M}_2^\natural$ *generalizes to a triple of* (*exo*) *flops*:



(80)

The map "*defo*" to the left is realized as follows: Deformations smooth $\mathcal{M}^\sharp$ by replacing the local cones $\mathbb{C}^4/Q$ centered at each node, $x_j^\sharp$, with a real 3-bundle over a copy of S^3. It is easy to see that a deformation of $G(s)$ from the non-transversal choice Eq. (15) to a transversal choice in Section 2.1 precisely induces the smoothing of the ground state variety from a (compactified) stratified variety of the type described in Section 2.2 to a smooth Calabi-Yau three-fold of the type described in Section 2.1. This provides the map $\mathcal{M}^\flat \leftrightarrow^{\mathrm{defo}} \bar{\mathcal{V}}$ in the diagram in Lemma 6.

Finally, we note that the above described homology of $\bar{\mathcal{V}}$ excluding the middle dimension, which we have not discussed herein, satisfies the requirements given in Ref. [17], and so is compatible with the 'mirror map'. The extension of this result to include the (co)homology groups in the middle dimension remains an open question for now and we hope to return to it in a future effort.

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